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Dynamic stiffness formulation and free vibration analysis of a three-layered sandwich beam

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Abstract

A dynamic stiffness theory of a three-layered sandwich beam is developed and subsequently used to investigate its free vibration characteristics. This is based on an imposed displacement field so that the top and bottom layers behave like Rayleigh beams, whilst the central layer behaves like a Timoshenko beam. Using Hamilton's principle the governing differential equations of motion of the sandwich beam are derived for the general case when the properties of each layer are dissimilar. For harmonic oscillation the solutions of these equations are found in exact analytical form, taking full advantage of the application of symbolic computation, which has also been used to obtain the amplitudes of axial force, shear force and bending moment in explicit analytical forms. The boundary conditions for responses and loads at both ends of the freely vibrating sandwich beam are then imposed to formulate the dynamic stiffness matrix, which relates harmonically varying loads to harmonically varying responses at the ends. Using the Wittrick–Williams algorithm the natural frequencies and mode shapes of some representative problems are obtained and discussed. The important degenerate case of a symmetric sandwich beam is also investigated.

Keywords: Dynamic stiffness method; Free vibration; Sandwich beam; Wittrick–Williams algorithm; Symbolic computation

1. Introduction

Sandwich construction offers the structural designer many attractive features, such as high specific stiffness, good buckling resistance, formability into complex shapes, easy reparability, and so on. Thus the analysis of such structural systems has been investigated–more or less continuously–for well over half a century. There are some excellent papers which contribute to the state-of-the-art, review earlier work and

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provide a long list of references on the subject, see for example, Frostig and Baruch (1994), Silverman (1995), Sainsbury and Zhang (1999) and Austin and Inman (2000).

By introducing visco-elasticity into the central element, good energy dissipation may be realised, but this paper does not deal with such non-conservative systems, but examines, in a way that has not been previously presented, the dynamical behaviour of an asymmetric three-layered element, namely a beam with three distinct components.

DiTaranto (1965), Mead and Sivakumaran (1966), Mead and Marcus (1969) and Mead (1982) are some of the earlier investigators who have examined the free vibration problems of sandwich beams using analytical methods, with due attention to the shearing that occurs between layers. Any Newtonian approach must take these shearing loads into account, with consequent complications in the analysis. Several other authors (Sakiyama et al., 1996; Fasana and Marchesiello, 2001; Banerjee, 2003) have also addressed this problem using an analytical model in which the top and bottom elements behave like beams in Bernoulli–Euler flexure, with the central element deforming only in transverse shear. Further developments that extend this model by adding a direct stress carrying capability to the central element can be found in He and Rao (1993), Bhimaraddi (1995) and Sisemore and Darvennes (2002). In many of these earlier works, assumptions such as the congruence of the top and bottom layers seriously restrict the value of the models.

The problems of modelling shears interacting between the layers can largely be side stepped by using an energy model, and thus implicitly, but not explicitly, representing the interaction. In particular, the use of Hamilton's principle allows a model to be developed in which the best possible elastic representation is achieved, subject to whatever restrictions are built into the analytical model. For example, a displacement field may be imposed which allows each element to behave in a relevant manner and as long as the representation can be justified, a good model will be realised.

This paper develops a model along these lines for the three-layered beam with no restrictions on the geometric and physical properties of each element. The three elements all have, in general, a mean axial (or longitudinal) motion as well as a common flexure and the system is fully coupled so that when the beam flexes, it has longitudinal response and vice versa. This model is of eighth order, and therein lies the difficulty in analytical development.

As is shown below, the completion of the analysis is achieved by using symbolic computation (Fitch, 1985; Rayna, 1986). This makes possible the development of a model in which the only approximations introduced are in the choice of the displacement field. The end product of the investigation is the development, and application, of the dynamic stiffness matrix of the three-layered beam. This retains all information derived by solving the governing differential equations subject to the appropriate boundary conditions. Thus the only source of error is in the choice of the displacement field.

The system of displacements used is as follows. All three layers have a common flexure. The top and bottom layers of the beam are assumed to bend in such a way that the cross-section rotates so as to be normal to the mid-plane flexure, as in the case of a Bernoulli–Euler beam, but with a longitudinal displacement and rotatory inertia taken into account. Thus the axial displacement varies linearly through the thickness. The central element also has a linear variation in axial displacement, but the cross-section does not rotate so as to be normal to the common flexure, and necessarily shears. This is modelled as a Timoshenko beam.

Such a procedure cannot generate a complete solution to the boundary value problem of the beam in vibration because it does not allow for variation in the transverse shear (and any associated non-planar bending). However, the likelihood is that the boundary zone between layers in which the shear changes rapidly is quite thin and so the inherent inaccuracy in the displacement field introduces only a small error into the energy expressions formulated, and so does not degrade significantly the whole model.

The resulting set of differential equations, which governs the free vibration of the three-layered beam, is of eighth order, which only degenerates into a simpler system of a sixth order flexure and second order longitudinal motion in the exceptional case when the top and bottom layers are identical. This is proved below.

A resume of the analysis developed is as follows. Using the assumed displacement field, the axial direct stresses, and any non-zero shear stresses, are evaluated, so that the expressions for the system kinetic and strain energies are obtained. The resulting Lagrangian is used in applying Hamilton's principle to obtain the partial differential equations and boundary conditions of the beam. By assuming harmonic motion, these partial differential equations are converted into total differential equations. These are further combined into a single eighth order total differential equation with constant coefficients. This is solved to generate the analytical solution to the problem. A state vector of loads (forces and moments) and a state vector of responses (displacements and rotations) are then expressed in terms of eight arbitrary constants of integration. On eliminating these constants the state vector of loads is expressed in terms of the state vector of response and this is the required dynamic stiffness matrix. From the outset the use of the symbolic computation package REDUCE (Fitch, 1985; Rayna, 1986) has enabled the Lagrangian to be formulated, the variational analysis to be undertaken, the governing differential equation and its solution, together with eighth order matrix manipulation, to be carried out. It is therefore, reasonable to ascertain that without resorting to a symbolic computation package such as REDUCE, the completion of the task of determining the dynamic stiffness matrix might well have been impossible.

The model developed assumes an isotropic material, and cannot readily be extended to represent fibre-reinforced laminated composite beams, as there would, in general, be a torsional response as well as the assumed motion for such materials. For orthotropic materials where no such restriction applies, the given analysis can readily be extended.

The theory has been programmed and applied to a number of cases, and the results are discussed and compared with published ones wherever possible. As the proposed method is exact (within the assumed displacement field), it can be used to provide a benchmark against which finite element and other studies may be validated. It is of particular value in comparing higher harmonic response, where finite element methods become less and less reliable as the mode number increases, whereas there is no corresponding loss of accuracy in the dynamic stiffness approach.

2. Theory

2.1. Derivation of the governing differential equations and solutions

In a rectangular Cartesian coordinate system Fig. 1 represents a three-layered sandwich beam with distinct element-1, 2 and 3 as shown. Each layer has its own geometric and material properties with a subscript denoting the layer number. Thus the top layer has thickness h_1, width b_1 and has a Young's modulus E_1,

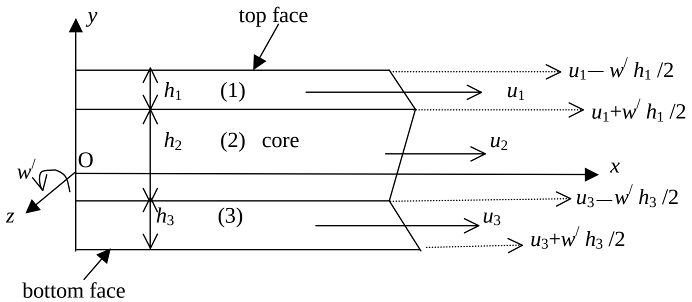

Fig. 1. The coordinate system and notation for a three-layered sandwich beam.

density ρ_1 cross-sectional area A_1 (=b_1h_1), and second moment of area I_1. Layers 1 and 3 do not shear, so their shear moduli are irrelevant, but layer 2 has shear modulus G_2.

Following the description of the chosen displacement field given in the introduction it is readily observed that the axial displacements vary linearly from $u_1 - \frac{h_1}{2}w'$ on the top to $u_1 + \frac{h_1}{2}w'$ at the interface between layers 1 and 2, and from $u_3 - \frac{h_3}{2}w'$ at the interface between layers 2 and 3 to $u_3 + \frac{h_3}{2}w'$ at the bottom of layer 3. Here, and later, primes denote partial differentiation with respect to x. The middle layer has a consequential shear as its imposed motion is determined by the continuity of layers, and an assumed linear variation across its thickness. (The mid-layer displacement u_2 in layer 2 is not further used.)

It follows that the axial strain varies

$$\text{from } u_1' - \frac{h_1}{2}w'' \text{ to } u_1' + \frac{h_1}{2}w'' \text{ in layer 1}$$

$$\text{from } u_1' + \frac{h_1}{2}w'' \text{ to } u_3' - \frac{h_3}{2}w'' \text{ in layer 2, and}$$

$$\text{from } u_3' - \frac{h_3}{2}w'' \text{ to } u_3' + \frac{h_3}{2}w'' \text{ in layer 3}$$

In addition layer 2 will have shearing strain

$$\gamma = w' + \frac{1}{h_2}\{u_1 - u_3 + (h_1 + h_3)w'/2\} = \frac{1}{h_2}(u_1 - u_3 + aw') \tag{1}$$

where

$$a = (h_1 + 2h_2 + h_3)/2 \tag{2}$$

Thus the entire strain system is known. For a linear isotropic material, the stresses and strain energy can now be formulated. In developing the strain and kinetic energies, repeated use is made of the following well known result.

If $f(x)$ is a linear function of x varying from $f_1 = f(x_1)$ at $x = x_1$ to $f_2 = f(x_2)$ at $x = x_2$ then

$$\int_{x_1}^{x_2} \{f(x)\}^2\,\mathrm{d}x = (x_2 - x_1)(f_1^2 + f_1f_2 + f_2^2)/3 \tag{3}$$

There are four sources of strain energy, three due to the axial strains in each layer and the fourth due to the shearing strain in layer 2. These have the following expressions

$$V_1 = \frac{1}{2}E_1A_1\int_0^L (u_1')^2\,\mathrm{d}x + \frac{1}{2}E_1I_1\int_0^L (w'')^2\,\mathrm{d}x \tag{4}$$

$$V_{2N} = \frac{1}{6}E_2A_2\int_0^L \left[(u_1')^2 + (u_1'u_3') + (u_3')^2 + \left(h_1 - \frac{h_3}{2}\right)u_1'w'' + \left(\frac{h_1}{2} - h_3\right)u_3'w'' + \frac{1}{4}(h_1^2 - h_1h_3 + h_3^2)(w'')^2\right]\mathrm{d}x \tag{5}$$

$$V_3 = \frac{1}{2}E_3A_3\int_0^L (u_3')^2\,\mathrm{d}x + \frac{1}{2}E_3I_3\int_0^L (w'')^2\,\mathrm{d}x \tag{6}$$

$$V_{2S} = \frac{1}{2}k_2A_2G_2\int_0^L \frac{(u_1 - u_3 + aw')^2}{h_2^2}\,\mathrm{d}x \tag{7}$$

where V_1 and V_3 are the strain energies of layers 1 and 3 due to normal strain and V_{2N} and V_{2S} are the strain energies of layer 2 due to normal and shearing strains respectively, and k_2 is the shear coefficient or shape factor (<1) which is introduced because the effective shear in layer 2 will not be equal to γ everywhere. Note that the bending properties of the core are properly represented although the term E_2I_2 does not appear explicitly in Eq. (5).

Thus the total strain energy $V = V_1 + V_{2N} + V_{2S} + V_3$ is

$$V = \frac{1}{2}\int_0^L \left[E_1A_1(u_1')^2 + E_1I_1(w'')^2 + \frac{E_2A_2}{3}\left\{(u_1')^2 + u_1'u_3' + (u_3')^2 + \left(h_1 - \frac{h_3}{2}\right)u_1'w''\left(\frac{h_1}{2} - h_3\right)u_3'w'' + \frac{1}{4}(h_1^2 - h_1h_3 + h_3^2)(w'')^2\right\} + E_3A_3(u_3')^2 + E_3I_3(w'')^2 + \frac{k_2A_2G_2}{h_2^2}(u_1 - u_3 + aw')^2\right]\mathrm{d}x \quad (8)$$

In a similar way, the kinetic energy T is the sum of four quantities representing the axial motion of each layer plus that due to transverse motion of the entire beam. Using an over dot to represent partial differentiation with respect to time, these are

$$T_1 = \frac{1}{2}\int_0^L \{m_1(\dot{u}_1)^2 + \rho_1I_1(\dot{w}')^2\}\,\mathrm{d}x \quad (9)$$

from layer 1,

$$T_2 = \frac{m_2}{6}\int_0^L \left[(\dot{u}_1)^2 + (\dot{u}_3)^2 + \dot{u}_1\dot{u}_3 + \left(h_1 - \frac{h_3}{2}\right)\dot{w}'\dot{u}_1 + \left(\frac{h_1}{2} - h_3\right)\dot{w}'\dot{u}_3 + \frac{1}{4}(h_1^2 - h_1h_3 + h_3^2)(\dot{w}')^2\right]\mathrm{d}x \quad (10)$$

from layer 2

$$T_3 = \frac{1}{2}\int_0^L \{m_3(\dot{u}_3)^2 + \rho_3I_3(\dot{w}')^2\}\,\mathrm{d}x \quad (11)$$

for layer 3, and

$$T_4 = \frac{1}{2}M\int_0^L (\dot{w})^2\,\mathrm{d}x \quad (12)$$

from the complete beam in transverse motion.

In Eqs. (9)–(12) $m_1 = \rho_1A_1$, $m_2 = \rho_2A_2$, $m_3 = \rho_3A_3$ and $M = m_1 + m_2 + m_3$ represent the mass per unit length for layers 1–3 and the complete beam respectively. Note that the rotatory inertia of the core has been accounted for although the term ρ_2I_2 does not appear explicitly in Eq. (10).

Thus $T = T_1 + T_2 + T_3 + T_4$ takes the following form

$$T = \frac{1}{2}\int_0^L \left[m_1\dot{u}_1^2 + \rho_1I_1(\dot{w}')^2 + \frac{m_2}{3}\left\{(\dot{u}_1)^2 + \dot{u}_1\dot{u}_3 + (\dot{u}_3)^2 + \left(h_1 - \frac{h_3}{2}\right)\dot{u}_1\dot{w}' + \left(\frac{h_1}{2} - h_3\right)\dot{u}_3\dot{w}' + \frac{1}{4}(h_1^2 - h_1h_3 + h_3^2)(\dot{w}')^2\right\} + m_3(\dot{u}_3)^2 + \rho_3I_3(\dot{w}')^2 + M\dot{w}^2\right]\mathrm{d}x \quad (13)$$

Combining T and V from Eqs. (13) and (8) the Lagrangian $\mathscr{L} = T - V$ takes the following form

$$\mathscr{L} = \frac{1}{2}\int_0^L \Big\{B_1\dot{u}_1^2 + 2B_6\dot{u}_1\dot{u}_3 + B_2\dot{u}_3^2 + 2B_5\dot{u}_1\dot{w}' + 2B_4\dot{u}_3\dot{w}'. + B_3(\dot{w}')^2 + M\dot{w}^2 - C_1(u_1')^2 - 2C_6u_1'u_3' - C_2(u_3')^2 - 2C_5u_1'w'' - 2C_4u_3'w'' - C_3(w'')^2 - C_7u_1^2 - 2C_{12}u_1u_3 - C_8u_3^2 - 2C_{11}u_1w' - 2C_{10}u_3w' - C_9(w')^2\Big\}\mathrm{d}x \quad (14)$$

where

$$B_1 = m_1 + \frac{m_2}{3}; \quad B_2 = m_3 + \frac{m_2}{3}; \quad B_3 = \rho_1 I_1 + \frac{m_2}{12}(h_1^2 - h_1 h_3 + h_3^2) + \rho_3 I_3 \tag{15}$$

$$B_4 = \frac{m_2}{6}\left(\frac{h_1}{2} - h_3\right); \quad B_5 = \frac{m_2}{6}\left(h_1 - \frac{h_3}{2}\right); \quad B_6 = \frac{m_2}{6} \tag{16}$$

$$C_1 = E_1 A_1 + \frac{E_2 A_2}{3}; \quad C_2 = \frac{E_2 A_2}{3} + E_3 A_3; \quad C_3 = E_1 I_1 + \frac{E_2 A_2}{12}(h_1^2 - h_1 h_3 + h_3^2) + E_3 I_3 \tag{17}$$

$$C_4 = \frac{E_2 A_2}{6}\left(\frac{h_1}{2} - h_3\right); \quad C_5 = \frac{E_2 A_2}{6}\left(h_1 - \frac{h_3}{2}\right); \quad C_6 = \frac{E_2 A_2}{6} \tag{18}$$

$$C_7 = \frac{k_2 A_2 G_2}{h_2^2}; \quad C_8 = C_7; \quad C_9 = a^2 C_7 \tag{19}$$

$$C_{10} = -a C_7; \quad C_{11} = -C_{10}; \quad C_{12} = -C_7 \tag{20}$$

with a given in Eq. (2).

Applying Hamilton's principle $\delta \int_{t_1}^{t_2} \mathscr{L}\, \mathrm{d}t = 0$ and using $\mathscr{L}$ from Eq. (14) there follows the set of differential equations

$$\left(-B_1\frac{\partial^2}{\partial t^2} + C_1\frac{\partial^2}{\partial x^2} - C_7\right)u_1 + \left(-B_6\frac{\partial^2}{\partial t^2} + C_6\frac{\partial^2}{\partial x^2} - C_{12}\right)u_3 + \left(-B_5\frac{\partial^2}{\partial t^2} + C_5\frac{\partial^2}{\partial x^2} - C_{11}\right)\frac{\partial w}{\partial x} = 0 \tag{21}$$

$$\left(-B_6\frac{\partial^2}{\partial t^2} + C_6\frac{\partial^2}{\partial x^2} - C_{12}\right)u_1 + \left(-B_2\frac{\partial^2}{\partial t^2} + C_2\frac{\partial^2}{\partial x^2} - C_8\right)u_3 + \left(-B_4\frac{\partial^2}{\partial t^2} + C_4\frac{\partial^2}{\partial x^2} - C_{10}\right)\frac{\partial w}{\partial x} = 0 \tag{22}$$

$$\begin{aligned}&\left(-B_5\frac{\partial^2}{\partial t^2} + C_5\frac{\partial^2}{\partial x^2} - C_{11}\right)\frac{\partial u_1}{\partial x} + \left(-B_4\frac{\partial^2}{\partial t^2} + C_4\frac{\partial^2}{\partial x^2} - C_{10}\right)\frac{\partial u_3}{\partial x} \\ &\quad + \left(-B_3\frac{\partial^4}{\partial x^2 \partial t^2} + C_3\frac{\partial^4}{\partial x^4} - C_9\frac{\partial^2}{\partial x^2} + M\frac{\partial^2}{\partial t^2}\right)w = 0\end{aligned} \tag{23}$$

Note the symmetry of the differential operators in Eqs. (21)–(23).

The associated boundary conditions generated by Hamilton's principle are as follows. The axial forces in layers 1 and 3 (F_1 and F_3) are

$$F_1 = -C_1\frac{\partial u_1}{\partial x} - C_6\frac{\partial u_3}{\partial x} - C_5\frac{\partial^2 w}{\partial x^2} \tag{24}$$

$$F_3 = -C_6\frac{\partial u_1}{\partial x} - C_2\frac{\partial u_3}{\partial x} - C_4\frac{\partial^2 w}{\partial x^2} \tag{25}$$

Note that each of the above two forces includes a contribution from layer 2.

The total shear force, S, in the direction Y, is given by

$$S = \left(-B_5\frac{\partial^2}{\partial t^2} + C_5\frac{\partial^2}{\partial x^2} - C_{11}\right)u_1 + \left(-B_4\frac{\partial^2}{\partial t^2} + C_4\frac{\partial^2}{\partial x^2} - C_{10}\right)u_3 + \left(-B_3\frac{\partial^2}{\partial t^2} + C_3\frac{\partial^2}{\partial x^2} - C_9\right)\frac{\partial w}{\partial x} \tag{26}$$

The total bending moment, M, acting on the beam is

$$M = -C_5 \frac{\partial u_1}{\partial x} - C_4 \frac{\partial u_3}{\partial x} - C_3 \frac{\partial^2 w}{\partial x^2} \tag{27}$$

For harmonic oscillation u_1, u_3 and w may be written in the following form

$$u_1 = U_1 e^{i\omega t}; \quad u_3 = U_3 e^{i\omega t}, \quad w = W e^{i\omega t} \tag{28}$$

where U_1, U_3 and W are the amplitudes of u_1, u_2 and w, ω is the circular (or angular) frequency of the free vibratory motion and $i = \sqrt{-1}$.

Eqs. (21)–(23) now take the following form as a result of the substitution of Eqs. (28).

$$\left(C_1 \frac{d^2}{dx^2} + B_1\omega^2 - C_7\right)U_1 + \left(C_6 \frac{d^2}{dx^2} + B_6\omega^2 - C_{12}\right)U_3 + \left(C_5 \frac{d^2}{dx^2} + B_5\omega^2 - C_{11}\right)\frac{dW}{dx} = 0 \tag{29}$$

$$\left(C_6 \frac{d^2}{dx^2} + B_6\omega^2 - C_{12}\right)U_1 + \left(C_2 \frac{d^2}{dx^2} + B_2\omega^2 - C_8\right)U_3 + \left(C_4 \frac{d^2}{dx^2} + B_4\omega^2 - C_{10}\right)\frac{dW}{dx} = 0 \tag{30}$$

$$\begin{aligned}&\left(C_5 \frac{d^2}{dx^2} + B_5\omega^2 - C_{11}\right)\frac{dU_1}{dx} + \left(C_4 \frac{d^2}{dx^2} + B_4\omega^2 - C_{10}\right)\frac{dU_3}{dx}\\ &\quad + \left\{C_3 \frac{d^4}{dx^4} + (B_3\omega^2 - C_9)\frac{d^2}{dx^2} - M\omega^2\right\}W = 0\end{aligned} \tag{31}$$

Introducing a non-dimensional length $\xi = x/L$ and writing $D = \frac{d}{d\xi}$, the above equations take the form

$$(C_1D^2 + \lambda_1)LU_1 + (C_6D^2 + \lambda_6)LU_3 + D(C_5D^2 + \lambda_5)W = 0 \tag{32}$$

$$(C_6D^2 + \lambda_6)LU_1 + (C_2D^2 + \lambda_2)LU_3 + D(C_4D^2 + \lambda_4)W = 0 \tag{33}$$

$$D(C_5D^2 + \lambda_5)LU_1 + D(C_4D^2 + \lambda_4)LU_3 + (C_3D^4 + \lambda_3D^2 - \lambda_7)W = 0 \tag{34}$$

where

$$\lambda_j = (\omega^2 B_j - C_{j+6})L^2 \tag{35}$$

for $j = 1, 2, 3, \ldots, 6$ and

$$\lambda_7 = M\omega^2 L^4 \tag{36}$$

By extensive algebraic manipulation the differential Eqs. (32)–(34) can be combined into a single eighth order differential equation satisfied by U_1, U_3 and W in the form

$$(D^8 + \alpha D^6 + \beta D^4 + \gamma D^2 + \delta)X = 0 \tag{37}$$

where X is one of U_1, U_3 or W.

Writing

$$\mu_0 = \frac{1}{3}\sum_{j=1}^{3}(\mu_j C_j + 2\mu_{j+3}C_{j+3}) \tag{38}$$

the coefficients α, β, γ and δ are given by

$$\alpha\mu_0 = \sum_{j=1}^{3}(\lambda_j\mu_j + 2\lambda_{j+3}\mu_{j+3}) \tag{39}$$

$$\beta\mu_0 = -\lambda_7\mu_3 + \sum_{j=1}^{3}(\mu_{j+6}C_j + 2\mu_{j+9}C_{j+3}) \tag{40}$$

$$\gamma\mu_0 = -(\lambda_1C_2 + \lambda_2C_1 - 2\lambda_6C_6)\lambda_7 + \frac{1}{3}\sum_{j=1}^{3}(\lambda_j\mu_{j+6} + 2\lambda_{j+3}\mu_{j+9}) \tag{41}$$

and

$$\delta\mu_0 = -\lambda_7\mu_9 \tag{42}$$

with

$$\mu_1 = C_2C_3 - C_4^2; \quad \mu_2 = C_3C_1 - C_5^2; \quad \mu_3 = C_1C_2 - C_6^2 \tag{43}$$

$$\mu_4 = C_5C_6 - C_1C_4; \quad \mu_5 = C_6C_4 - C_2C_5; \quad \mu_6 = C_4C_5 - C_3C_6 \tag{44}$$

$$\mu_7 = \lambda_2\lambda_3 - \lambda_4^2; \quad \mu_8 = \lambda_3\lambda_1 - \lambda_5^2; \quad \mu_9 = \lambda_1\lambda_2 - \lambda_6^2 \tag{45}$$

$$\mu_{10} = \lambda_5\lambda_6 - \lambda_1\lambda_4; \quad \mu_{11} = \lambda_6\lambda_4 - \lambda_2\lambda_5; \quad \mu_{12} = \lambda_4\lambda_5 - \lambda_3\lambda_6 \tag{46}$$

The differential Eq. (37) is linear with constant coefficients so that the solution is sought in the form

$$X = X_0\mathrm{e}^{r\xi} \tag{47}$$

Substituting Eq. (47) into Eq. (37) yields the auxiliary equation

$$r^8 + \alpha r^6 + \beta r^4 + \gamma r^2 + \delta = 0 \tag{48}$$

The above equation is a quartic in $p = r^2$ namely

$$p^4 + \alpha p^3 + \beta p^2 + \gamma p + \delta = 0 \tag{49}$$

which may be solved in a routine way.

Some pair or pairs of complex roots may occur, but as U_1, U_3 and W are all real, the associated coefficient, say X_j, in the solution for $X = \sum_{j=1}^{8}X_j\mathrm{e}^{r_j\xi}$ will also be complex. As complex roots occur only in conjugate pairs, the associated X_j will also occur in conjugate pairs.

Thus, the solution for U_1, U_3 and W can be written as

$$U_1(\xi) = \sum_{j=1}^{8}P_j\mathrm{e}^{r_j\xi}; \quad U_3(\xi) = \sum_{j=1}^{8}Q_j\mathrm{e}^{r_j\xi}; \quad W(\xi) = \sum_{j=1}^{8}R_j\mathrm{e}^{r_j\xi} \tag{50}$$

where r_j ($j = 1,2\ldots,8$) are the eight roots of the auxiliary equation and P_j, Q_j and R_j, ($j = 1,2\ldots,8$) are three sets of eight, possibly complex, constants.

The rotation of the cross-section $\Theta(\xi)$ can be expressed as

$$\Theta(\xi) = \frac{1}{L}\frac{\mathrm{d}W}{\mathrm{d}\xi} = \frac{1}{L}\sum_{j=1}^{8}r_jR_j\mathrm{e}^{r_j\xi} \tag{51}$$

By substituting Eqs. (50) into Eqs. (32)–(34) it can be shown that the constants P_j and Q_j are related to R_j as follows so that the responses U_1, U_3, W and Θ are all linear combinations of R_j.

$$P_j = f_j R_j \tag{52}$$

and

$$Q_j = g_j R_j \tag{53}$$

where

$$f_j = r_j\{(C_2 r_j^2 + \lambda_2)(C_5 r_j^2 + \lambda_5) - (C_4 r_j^2 + \lambda_4)(C_6 r_j^2 + \lambda_6)\}/\Delta_j \tag{54}$$

and

$$g_j = r_j\{(C_1 r_j^2 + \lambda_1)(C_4 r_j^2 + \lambda_4) - (C_5 r_j^2 + \lambda_5)(C_6 r_j^2 + \lambda_6)\}/\Delta_j \tag{55}$$

with

$$\Delta_j = L\{(C_6 r_j^2 + \lambda_6)^2 - (C_1 r_j^2 + \lambda_1)(C_2 r_j^2 + \lambda_2)\} \tag{56}$$

The expressions for the amplitudes of axial forces in layers 1 and 3 (F_1 and F_3), the shear force across the cross-section (S) and the bending moment in the cross-section (M) follow from Eqs. (24)–(27). Noting that these forces and moment vary harmonically during the vibratory motion in the same way as the displacements, so that they are (in terms of the non-dimensional variable $\xi = x/L$) given by

$$F_1(\xi) = -\frac{C_1}{L}\left(\frac{dU_1}{d\xi} + \frac{C_6}{C_1}\frac{dU_3}{d\xi} + \frac{C_5}{C_1 L}\frac{d^2W}{d\xi^2}\right) \tag{57}$$

$$F_3(\xi) = -\frac{C_2}{L}\left(\frac{C_6}{C_2}\frac{dU_1}{d\xi} + \frac{dU_3}{d\xi} + \frac{C_4}{C_2 L}\frac{d^2W}{d\xi^2}\right) \tag{58}$$

$$S(\xi) = \frac{C_3}{L^3}\left(\frac{\lambda_5 L}{C_3}U_1 + \frac{C_5 L}{C_3}\frac{d^2U_1}{d\xi^2} + \frac{\lambda_4 L}{C_3}U_3 + \frac{C_4 L}{C_3}\frac{d^2U_3}{d\xi^2} + \frac{\lambda_3}{C_3}\frac{dW}{d\xi} + \frac{d^3W}{d\xi^3}\right) \tag{59}$$

$$M(\xi) = -\frac{C_3}{L^2}\left(\frac{C_5 L}{C_3}\frac{dU_1}{d\xi} + \frac{C_4 L}{C_3}\frac{dU_3}{d\xi} + \frac{d^2W}{d\xi^2}\right) \tag{60}$$

With the help of Eqs. (50), (52) and (53), Eqs. (57)–(60) can be written as

$$F_1(\xi) = -\frac{C_1}{L}\left(\sum_{j=1}^{8} r_j\left(f_j + \frac{C_6}{C_1}g_j + \frac{C_5}{C_1 L}r_j\right)R_j e^{r_j\xi}\right) \tag{61}$$

$$F_3(\xi) = -\frac{C_2}{L}\left(\sum_{j=1}^{8} r_j\left(\frac{C_6}{C_2}f_j + g_j + \frac{C_4}{C_2 L}r_j\right)R_j e^{r_j\xi}\right) \tag{62}$$

$$S(\xi) = \frac{C_3}{L^3}\left(\sum_{j=1}^{8}\left(\frac{\lambda_5 L}{C_3}f_j + \frac{C_5 L}{C_3}r_j^2 f_j + \frac{\lambda_4 L}{C_3}g_j + \frac{C_4 L}{C_3}r_j^2 g_j + r_j^3 + \frac{\lambda_3}{C_3}r_j\right)R_j e^{r_j\xi}\right) \tag{63}$$

$$M(\xi) = -\frac{C_3}{L^2}\left(\sum_{j=1}^{8} r_j\left(\frac{C_5 L}{C_3}f_j + \frac{C_4 L}{C_3}g_j + r_j\right)R_j e^{r_j\xi}\right) \tag{64}$$

where the loads $F_1(\xi)$, $F_3(\xi)$, $S(\xi)$ and $M(\xi)$ are also linear combinations of R_j.

2.2. Formulation of the dynamic stiffness matrix

The amplitudes of the responses and loads of the freely vibrating sandwich beam are given by Eqs. (50), (51) and (61)–(64), respectively which can now be related by the dynamic stiffness matrix on eliminating the arbitrary constants R_j ($j = 1, 2, 3, \ldots, 8$).

Referring to Fig. 2, the boundary conditions for responses and loads of the sandwich beam are as follows.

At the left hand end, $\xi = 0$ ($x = 0$), the responses are $U_1(0)$, $U_3(0)$, $W(0)$ and $\Theta(0)$. The corresponding responses at the right hand end, $\xi = 1(x = L)$, are $U_1(1)$, $U_3(1)$, $W(1)$ and $\Theta(1)$, see Fig. 2. By substituting $\xi = 0$ and $\xi = 1$ in Eqs. (50) and (51), these boundary conditions give

$$U_1(0) = \sum_{j=1}^{8} P_j; \quad U_3(0) = \sum_{j=1}^{8} Q_j; \quad W(0) = \sum_{j=1}^{8} R_j; \quad \Theta(0) = \frac{1}{L}\sum_{j=1}^{8} r_j R_j \tag{65a}$$

$$U_1(1) = \sum_{j=1}^{8} P_j e^{r_j}; \quad U_3(1) = \sum_{j=1}^{8} Q_j e^{r_j}; \quad W(1) = \sum_{j=1}^{8} R_j e^{r_j}; \quad \Theta(1) = \frac{1}{L}\sum_{j=1}^{8} r_j R_j e^{r_j} \tag{65b}$$

Eqs. (65a) and (65b) can be written in the following matrix form and by using Eqs. (52) and (53) and simply referring the state vector of response $U_1(0)$, $U_3(0)$, $W(0)$, $\Theta(0)$, $U_1(1)$, $U_3(1)$, $W(1)$ and $\Theta(1)$, to only one set of arbitrary constants R_j as follows.

$$\begin{bmatrix} U_1(0) \\ U_3(0) \\ W(0) \\ \Theta(0) \\ U_1(1) \\ U_3(1) \\ W(1) \\ \Theta(1) \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ r_1/L & r_2/L & r_3/L & r_4/L & r_5/L & r_6/L & r_7/L & r_8/L \\ f_1 e^{r_1} & f_2 e^{r_2} & f_3 e^{r_3} & f_4 e^{r_4} & f_5 e^{r_5} & f_6 e^{r_6} & f_7 e^{r_7} & f_8 e^{r_8} \\ g_1 e^{r_1} & g_2 e^{r_2} & g_3 e^{r_3} & g_4 e^{r_4} & g_5 e^{r_5} & g_6 e^{r_6} & g_7 e^{r_7} & g_8 e^{r_8} \\ e^{r_1} & e^{r_2} & e^{r_3} & e^{r_4} & e^{r_5} & e^{r_6} & e^{r_7} & e^{r_8} \\ r_1 e^{r_1}/L & r_2 e^{r_2}/L & r_3 e^{r_3}/L & r_4 e^{r_4}/L & r_5 e^{r_5}/L & r_6 e^{r_6}/L & r_7 e^{r_7}/L & r_8 e^{r_8}/L \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} \tag{66}$$

or

$$\boldsymbol{\delta} = \mathbf{B}\mathbf{R} \tag{67}$$

where the displacement and constant vectors $\boldsymbol{\delta}$ and $\mathbf{R}$ and the square matrix $\mathbf{B}$ follow from Eq. (66).

Similarly at the left hand end, $\xi = 0$ ($x = 0$), the loads are $F_1(0)$, $F_3(0)$, $S(0)$ and $M(0)$, and the corresponding loads at the right hand end, $\xi = 1$ ($x = L$), are $F_1(1)$, $F_3(1)$, $S(1)$ and $M(1)$, see Fig. 2. By substituting $\xi = 0$ and 1 in Eqs. (61)–(64), and noting that the signs for the forces must be reversed at the right

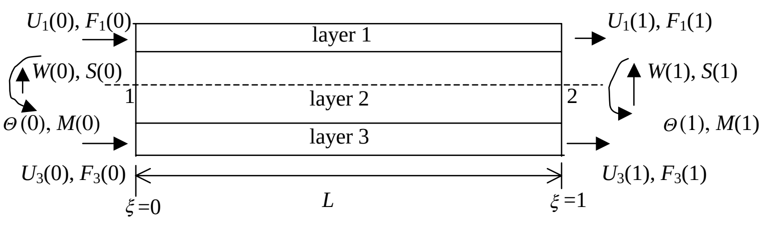

Fig. 2. End conditions for responses and loads for the three-layered sandwich beam.

hand end as a consequence of the sign convention, these boundary conditions give the following matrix relationship.

$$\begin{bmatrix} F_1(0) \\ F_3(0) \\ S(0) \\ M(0) \\ F_1(1) \\ F_3(1) \\ S(1) \\ M(1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} \tag{68}$$

or

$$\mathbf{F} = \mathbf{AR} \tag{69}$$

where **F** is the state vector of loads and the elements of **A** are as follows.

$$a_{1j} = -\frac{C_1}{L} r_j \left(f_j + \frac{C_6}{C_1} g_j + \frac{C_5}{C_1 L} r_j \right) \tag{70}$$

$$a_{2j} = -\frac{C_2}{L} r_j \left(\frac{C_6}{C_2} f_j + g_j + \frac{C_4}{C_2 L} r_j \right) \tag{71}$$

$$a_{3j} = \frac{C_3}{L^3} \left(\frac{\lambda_5 L}{C_3} f_j + \frac{C_5 L}{C_3} f_j r_j^2 + \frac{\lambda_4 L}{C_3} g_j + \frac{C_4 L}{C_3} g_j r_j^2 + \frac{\lambda_3}{C_3} r_j + r_j^3 \right) \tag{72}$$

$$a_{4j} = -\frac{C_3}{L^2} r_j \left(\frac{C_5 L}{C_3} f_j + \frac{C_4 L}{C_3} g_j + r_j \right) \tag{73}$$

$$a_{5j} = \frac{C_1}{L} r_j \left(f_j + \frac{C_6}{C_1} g_j + \frac{C_5}{C_1 L} r_j \right) \mathrm{e}^{r_j} \tag{74}$$

$$a_{6j} = \frac{C_2}{L} r_j \left(\frac{C_6}{C_2} f_j + g_j + \frac{C_4}{C_2 L} r_j \right) \mathrm{e}^{r_j} \tag{75}$$

$$a_{7j} = -\frac{C_3}{L^3} \left(\frac{\lambda_5 L}{C_3} f_j + \frac{C_5 L}{C_3} f_j r_j^2 + \frac{\lambda_4 L}{C_3} g_j + \frac{C_4 L}{C_3} g_j r_j^2 + \frac{\lambda_3}{C_3} r_j + r_j^3 \right) \mathrm{e}^{r_j} \tag{76}$$

$$a_{8j} = \frac{C_3}{L^2} r_j \left(\frac{C_5 L}{C_3} f_j + \frac{C_4 L}{C_3} g_j + r_j \right) \mathrm{e}^{r_j} \tag{77}$$

where $j = 1, 2, 3, \ldots, 8$.

The dynamic stiffness matrix can now be formulated by eliminating the vector of constants **R** from the Eqs. (67) and (69) to give

$$\mathbf{F} = \mathbf{AB}^{-1} \boldsymbol{\delta} = \mathbf{K} \boldsymbol{\delta} \tag{78}$$

where

$$\mathbf{K} = \mathbf{AB}^{-1} \tag{79}$$

is the required dynamic stiffness matrix.

2.3. Further observations and reduction of the theory to the symmetric case

An alternative formulation of the governing equations of motion and boundary conditions introducing new displacements, say U_4 and U_5, which combine U_1 and U_3 provides a further insight into the problem as follows.

Writing $2U_4 = U_1 + U_3$ and $2U_5 = U_1 - U_3$ so that $U_1 = U_4 + U_5$ and $U_3 = U_4 - U_5$. Eqs. (32)–(34) can be recast in terms of U_4 and U_5. These equations are not explicitly given as new equations of interest, but are found by adding and subtracting Eqs. (32) and (33) respectively, and then substituting for U_1 and U_3 in terms of U_4 and U_5. Eq. (34) is reformed by direct substitution of U_1 and U_3. After some manipulation, the revised set of equations is

$$\{(C_1 + C_2 + 2C_6)D^2 + \lambda_1 + \lambda_2 + 2C_6\}LU_4 + \{(C_1 - C_2)D^2 + \lambda_1 - \lambda_2\}LU_5 + D\{(C_4 + C_5)D^2 + \lambda_4 + \lambda_5\}W = 0 \quad (80)$$

$$\{(C_1 - C_2)D^2 + \lambda_1 - \lambda_2\}LU_4 + \{(C_1 + C_2 - 2C_6)D^2 + \lambda_1 + \lambda_2 - 2\lambda_6\}LU_5 + D\{(C_5 - C_4)D^2 + \lambda_5 - \lambda_4\}W = 0 \quad (81)$$

$$D\{(C_4 + C_5)D^2 + \lambda_4 + \lambda_5\}LU_4 + D\{(C_5 - C_4)D^2 + \lambda_5 - \lambda_4\}LU_5 + (C_3D^4 + \lambda_3D^2 - \lambda_7)W = 0 \quad (82)$$

This alternative formulation appears—at first sight—merely to complicate the presentation. However, when layers 1 and 3 are identical in geometric and material properties, an important simplification occurs.

In this case

$$B_1 = B_2 = m_1 + \frac{m_2}{3}; \quad B_4 = -B_5 = -\frac{m_2 h_1}{12} \quad (83)$$

and

$$C_1 = C_2 = E_1A_1 + \frac{E_2A_2}{3}; \quad C_4 = -C_5 = -\frac{E_2A_2h_1}{12} \quad (84)$$

Clearly $\lambda_1 - \lambda_2 = 0$ and $\lambda_4 + \lambda_5 = 0$ for this case as it can be seen from Eqs. (18)–(20) and (35) that for this case $\lambda_1 = \lambda_2$ and $\lambda_4 = -\lambda_5$.

Hence the first equation involves U_4 only, whilst the other two do not involve U_4. Thus the system decouples.

There are a number of research papers which deals with symmetric sandwich beams and some start with the assumption that $U_1 = -U_3$, or $U_4 = 0$, see Mead and Sivakumaran (1966) and Mead (1982). It is clear that the general case splits into two separate systems, one purely longitudinal with no flexure, and the other flexural, with a push-pull action of the inner and outer layers, but with no mean longitudinal motion. The only circumstance in which an exception to this rule can occur is when both decoupled systems have a common eigenvalue, in which case the normal modes may include a mixture of the two motions.

For the general system of eighth order, the special case, where layers 1 and 3 are identical, can be approached by a limiting analysis in which the discrepancy between material and geometric properties is governed by a parameter that may be allowed to tend to zero. Let, for example, $E_3A_3 = E_1A_1(1 + \varepsilon)$ and let all other properties of the top and bottom layers be similarly related. Thus as ε tends to zero, the two layers become identical.

Now the eigenvalues of the general eighth order system may be assumed distinct (the exceptional case is explained later). Then the eigenvectors for small ε will weakly couple the eigenvectors of the purely longitudinal case, and the combined push-pull longitudinal and flexural motion case. For an eigenvalue later to be identified with an eigenvalue of the push-pull longitudinal with flexure case, the eigenvector component of U_4 will go to zero with ε. Thus this physical system cannot support the possibility of $U_4 \neq 0$. This shows that the assumption that—in the symmetric case, $U_1 = -U_3$ (or $U_4 = 0$) is not only justified, but is essential.

If the two systems described above have a common eigenvalue, then an eigenvector of one system is not suppressed when the other system operates. This is the exceptional case mentioned above, and leads to an eigenvector with U_1 and U_3 arbitrarily related.

One further case should be mentioned, when the material properties are the same for all three layers, and for any h_i, the model does not reduce to a Bernoulli–Euler beam, as the layer 2 is also effective in shear.

3. Application of the dynamic stiffness matrix

The dynamic stiffness matrix can now be used to compute the natural frequencies and mode shapes of either a single three-layered sandwich beam or an assembly of such beams, for example, a continuous sandwich beam on multiple supports. An accurate and reliable method of calculating the natural frequencies and mode shapes is to apply the dynamic stiffness matrix using with the well-known algorithm of Wittrick and Williams (1971), which has featured in numerous papers (Williams and Wittrick, 1983; Williams, 1993). The algorithm, unlike its proof, is very simple to use but for a detailed insight interested readers are referred to the original work of Wittrick and Williams (1971). Essentially the algorithm needs the dynamic stiffness matrices of individual members such as the three-layered beams considered in this paper together with other members in a structure and information about their natural frequencies when both ends are clamped. This information is needed to ensure that no natural frequencies of the structure are missed. The zeroes of the determinant of the matrix **B** in Eq. (67) give the clamped–clamped natural frequencies of the three-layered beam. It should be noted that the actual requirement for the algorithm is to isolate these clamped–clamped natural frequencies (that is to determine how many such natural frequencies are there below a specified trial frequency) rather than actually calculating them. The Wittrick–Williams algorithm in essence gives the number of natural frequencies of a structure that exists below an arbitrarily chosen trial frequency rather than actually determining them. This simple feature of the algorithm can be used to calculate any natural frequency of the structure to any desirable accuracy.

4. Scope, limitations and pitfalls of the theory

This paper presents the free vibration theory of a three-layered sandwich beam by developing its dynamic stiffness matrix. The governing differential equations of motion have been formed using Hamilton's principle. The entire formulation is based on a number of assumptions that may limit the applications of the theory. The main assumptions are: (1) all three layers of the beam are elastic, (2) the shearing strains in the top and bottom layers are negligible and therefore, they posses almost infinite shear rigidity whereas the middle layer or core is shear deformable and thus has a finite shear rigidity, (3) there is no slippage between any of the layers, (4) the shearing strain is constant across the depth of the middle layer, and (5) transverse direct strains in all three layers are negligible.

The theory presented is based on the premises that the top and bottom layers of the beam are individual Bernoulli–Euler beams coupled together by the core. The material and geometric properties of the three layers may vary markedly, and for some exceptional problems, the exponential terms in the solution with positive real parts may become very large, presenting numerical difficulties. The problem can be overcome by re-scaling of the constants calculated from the geometric and material properties of the beam. One of the strategies that can be adopted is as follows. Before resorting to any matrix inversion operation the constants associated with positive real parts can be modified by first multiplying and later dividing by a suitably chosen exponential term, say, $e^{-\lambda}$ where λ can be typically the length L of the beam.

It should also be noted that for a symmetric three-layered sandwich beam (when the top and bottom faces are identical) unless the dynamic stiffness matrix is developed using the procedure given in Section 2.3, the general theory for the asymmetric beam will fail. This is because the coefficient matrix of the governing equations becomes singular. However, if the top and bottom layers are in fact identical, the solution using the general theory can still be obtained if the properties of the top and bottom layers are changed by a very small amount. For example, by changing the thickness of one of the two identical layers by a negligibly small amount may permit a solution. However, care should be exercised because exceptionally small difference of less than say, of the order of 10^{-8} may cause numerical ill conditioning or give completely wrong answers.

5. Results and discussions

There are a few research papers (Frostig and Baruch, 1994; Sainsbury and Zhang, 1999; Bhimaraddi, 1995) which provide theory for a three-layered beam of unequal thicknesses, but unfortunately, when presenting numerical results, these papers have resorted to symmetric sandwich beams for which the top and bottom layers have identical properties. The authors were unable to find specimen results for natural frequencies and mode shapes of a three-layered sandwich beam for which all of the three layers have different geometric and material properties. However, they have made every possible effort to validate their theory, particularly by making sure that the limiting case when a uniform solid beam is divided into three fictitious layers the theory gives correct results. The theory has also been checked when the thickness and other properties of the top and bottom layers of the beam approach equal values resulting in a symmetric three-layered beam.

For illustrative purposes numerical results for three examples are provided. The first is a three-layered sandwich beam of rectangular cross-section and length 0.5 m for which the top and bottom layers have thicknesses 2 mm and 3 mm respectively, and they are made of steel whereas the middle layer is of rubber material with thickness 20 mm. Each of the three layers has the same width of 40 mm. The properties used for steel (with suffix s) and rubber (with suffix r) are $E_s = 210\,\text{GPa}$, $G_s = 80\,\text{GPa}$, $\rho_s = 7850\,\text{kg/m}^3$ and $E_r = 1.5\,\text{MPa}$, $G_r = 0.5\,\text{MPa}$, $\rho_r = 950\,\text{kg/m}^3$ respectively. The shear correction factor (shape factor) for the cross-section is taken to be 2/3. The first four natural frequencies and mode shapes of the beam with cantilever end conditions are illustrated in Fig. 3 which show that the modes are all flexural involving transverse displacements W only. This is to be expected because of the large extensional stiffnesses of the two faces. Clearly the modes involving axial displacements U_1 and U_3 are unlikely to occur in the lower frequency range for such problems.

In the second example, the rubber core of the above problem is replaced by lead keeping the rest of the structure unaltered. The properties used for lead (with suffix l) are $E_l = 16\,\text{GPa}$, $G_l = 5.5\,\text{GPa}$ and $\rho_l = 11\,100\,\text{kg/m}^3$. The first four natural frequencies and mode shapes of the cantilever three-layered beam are shown in Fig. 4, which reveals some interesting features. The first mode is basically a flexural mode of the beam involving bending displacement W only. Although the second and third modes are dominated by flexural displacement W, they are nevertheless, coupled modes which show some amount of coupling with the axial displacements of the top and bottom layers U_1 and U_3 respectively, but acting in opposite directions. Interestingly, the fourth mode is a pure axial mode with displacements U_1 and U_3 in the same direction with no amount of flexural displacements present.

The third and the final example is that of a symmetric sandwich beam with cantilever end conditions for which results have been reported by a Mead and Sivakumaran (1966) and Ahmed (1972). The

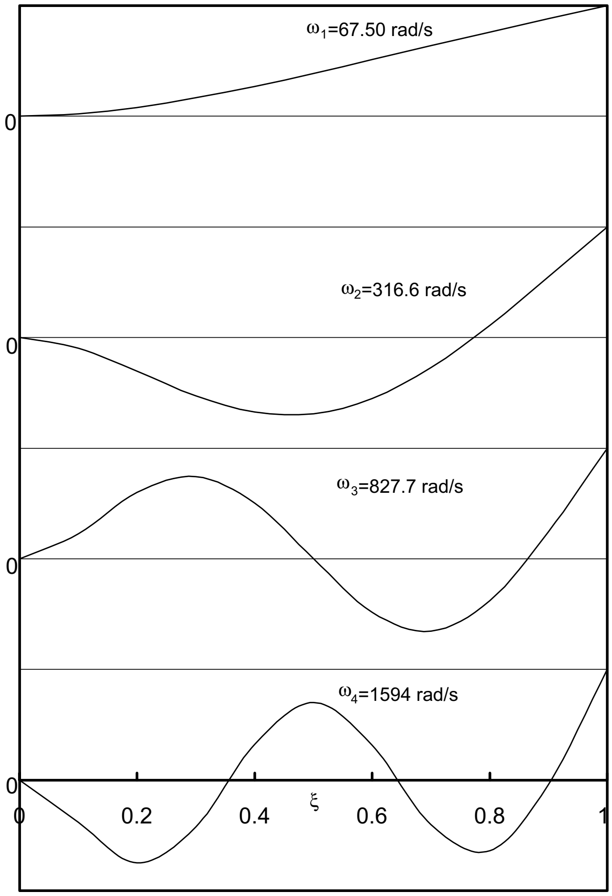

Fig. 3. Natural frequencies and mode shapes of the three-layered sandwich beam of example 1 (—) *W*.

geometrical dimensions and material properties of the beam are as follows. The length $L = 0.7112\,\text{m}$, core thickness $h_2 = 12.7\,\text{mm}$, the face thicknesses $h_1 = h_3 = 0.4572\,\text{mm}$, face Young's modulus $E_1 = E_3 = 69$ GPa, face density $\rho_1 = \rho_3 = 2680\,\text{kg/m}^3$, core Young's modulus $E_2 = 0.215\,\text{GPa}$, core shear modulus $G_2 = 82.8\,\text{MPa}$, core density $\rho_2 = 32.8\,\text{kg/m}^3$. In order to make the results directly comparable with published literature the mass of the core was taken into account, but its axial and bending stiffnesses were ignored and thus making it deform only in shear. The first four natural frequencies using the present theory are shown in Table 1 along side the results of Mead and Sivakumaran (1966) and Ahmed (1972). The maximum discrepancy between the results from present theory and the published results is around 2% as can be seen. Some of these small discrepancies may be attributed to the differences in the rigidity and mass data used in the analysis. When the axial and bending stiffnesses of the core together with its mass and shear stiffnesses are all taken into account the four natural frequencies become 36.74 Hz, 216.2 Hz, 555.7 Hz and 982.6 Hz, respectively. A comparison of these results with the ones quoted in Table 1 indicates that the inclusion of the axial and bending stiffnesses of the core increases the natural frequencies by around 8%. The axial and bending stiffnesses of the core have relatively small effect on the natural frequencies of this particular sandwich beam and thus the simpler theories used by previous investigators seem to be justified for such problems.

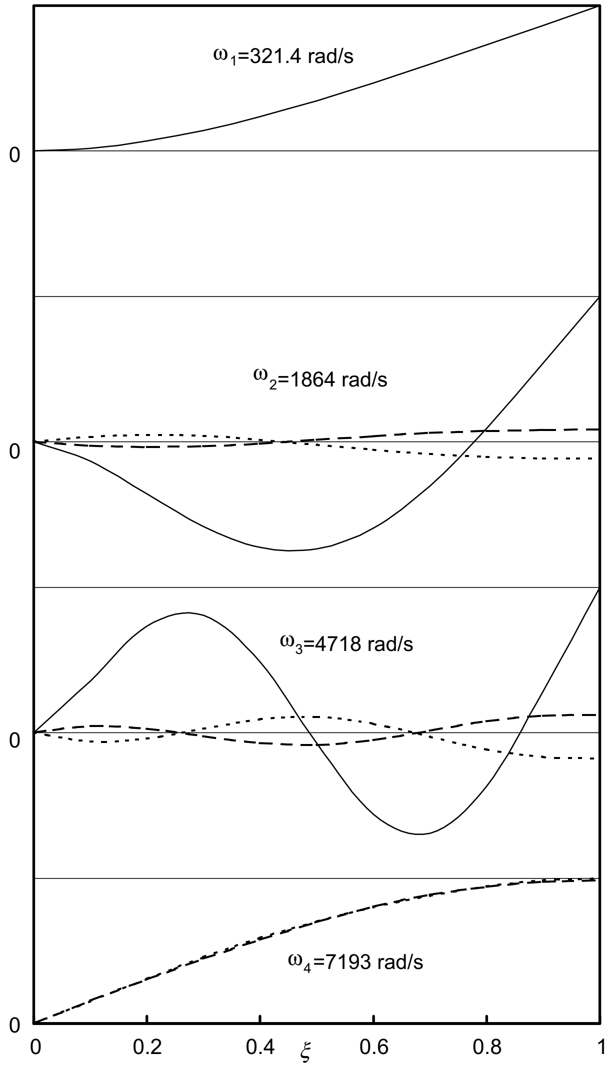

Fig. 4. Natural frequencies and mode shapes of the three-layered sandwich beam of example 2 (- - -) U_1; (– – –) U_3; (—) W.

Table 1
Natural frequencies of a symmetric sandwich beam with canliver end conditions

Frequency no.	Natural frequencies (Hz)		
	Mead and Sivakumaran (1966) (see Table IV, Column 1)	Ahmed (1972) (see Table 3, Column 4)	Present theory
1	34.24	33.97	33.74
2	201.9	200.5	198.8
3	520.9	517.0	511.4
4	925.4	918.0	905.1

6. Conclusions

Starting with a displacement field that assumes that the axial displacements vary linearly in each element, and that all the layers have a common flexure, the governing differential equations of motion in free vibration of an asymmetric three-layered beam are developed and used to obtain the dynamic stiffness matrix, which relates harmonically varying nodal loads with harmonically varying nodal responses. This has been supported by extensive use of symbolic computation. The application of the dynamic stiffness matrix is demonstrated to obtain numerical results for three examples. Some of these results are compared with published results that show good agreement. The development of the theory presented here demonstrate great potential for the use of symbolic computation in advanced structural analysis and is expected to pave the way for further research on the dynamic stiffness formulation of more complex structural elements.

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